A PROBLEM OF ALLOCATION OF SUPPORTING FIRE IN COMBAT AS A ZERO SUM DIFFERENTIAL GAME

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THESIS

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ABSTRACT

Optimal fire-support strategies are studied through a deterministic differential game using Lanchester-type equations of warfare. In addition to the MAX-MIN principle, the theory of singular extremals is required to solve this prescribed duration combat problem. The combat is between two heterogeneous forces, each composed of infantry and a supporting weapon system (artillery). In contrast to previous work reported in the literature, the attrition structure of the problem at hand leads to the optimal fire-support strategy of the attacker requiring him to sometimes split his artillery fire between enemy infantry and artillery (counterbattery fire). Numerical examples are given. The military significance (based on the marginal value interpretation of the dual variables) of various optimality conditions is discussed.

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TABLE I. Notation

$x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ -- State variables, number of survivors of X_1, X_2, Y_1, Y_2 at time t.

T -- Specified duration of the battle.

$p_1(t)$, $p_2(t)$, $q_1(t)$, $q_2(t)$ -- Adjoint (dual) variables, the value of X_1, X_2, Y_1, Y_2, respectively at time t.

a_1 -- Attrition coefficient, average rate at which one Y_1 attrits X_1.

a_2 -- Average rate at which one Y_2 attrits X_1.

b_1 -- Average rate at which one X_2 attrits Y_1.

b_2 -- Average rate at which one X_2 attrits Y_2.

c -- Rate at which Y_2 attrits X_2.

ϕ -- Decision variable, fraction of Y_2 fire directed at X_1, $0 \leq \phi \leq 1$.

ψ -- Fraction of X_2 fire directed at Y_1, $0 \leq \psi \leq 1$.

ϕ^*, ψ^* -- Optimal values of ϕ and ψ.

J -- Criterion functional, the ratio of the final values of X_1 and Y_1, $J = \dfrac{X_1(T)}{Y_1(T)}$.

τ -- Backwards time, $\tau = T - t$.

τ_ϕ -- Backward time at which strategy change occurs for Y.

τ_ψ -- Backward time at which strategy change occurs for X.

$\tau_\psi{}^*$ -- Backward time at which X_2 begins to split his fire.

x_1f, x_2f, y_1f, y_2f -- Final values of the state variables at $t = T$.

$\dfrac{y_1f^*}{y_2f^*} = \dfrac{a_2b_2}{a_1b_1} e^{-b_2x_2f\tau_\psi{}^*}$, The value of this ratio which insures a singular solution in ψ^*.

I. INTRODUCTION

The determination of optimal fire distribution strategies for supporting weapon systems such as artillery, air resources, or naval gunfire is a major problem of military operations research. This problem in one form or another is probably one of the most extensively studied problems in both the open literature and in classified sources. The problem is of interest to the military tactician in hopes of gaining a clearer understanding of the circumstances under which enemy infantry should be engaged by a supporting weapon system and those under which "counter-battery" fire is to be preferred. It should be stressed that the objective in researching this problem is not to determine what strategies <u>are</u> employed by combatants, but rather to attempt to understand what strategies <u>should</u> be employed for optimal effectiveness.

It is appropriate to briefly sketch past efforts in the study of "allocation" problems and how this work is related to those efforts. The problem of the appropriate mixture of tactical and strategic forces (another aspect of the optimal fire support strategy problem) was extensively debated by experts during World War II. Some analysis details may be found in the classical book by Morse and Kimball [7]. The problem was studied at Rand

in the late 1940's and early 1950's in the form of a tactical air-war [2]. It would probably not be unrealistic to claim that this problem stimulated early research on both dynamic programming [1] and on differential games [2], [3]. Today the problem of optimal air-war strategies is being extensively investigated by a number of organizations (see, for example, [6], [8], [14]).

Another related problem was considered by Weiss [15], who studied the optimal selection of targets for engagements by a supporting weapon system. Later Taylor [11] justified the optimality of strategies determined by Weiss. Recently, Kawara [5] studied optimal strategies for supporting weapon systems in an attack scenario which is a variation of the model that was considered by Weiss [15]. Other recent work has considered various conceptual and computational aspects of the time-sequential combat games [8], [9], [10].

Since the work here may be considered to be an elaboration upon and extension of Kawara's fire support differential game [5], it seems appropriate to review the major results of that work and to relate this work to it. In [5] Kawara considered combat between two heterogeneous forces, each composed of infantry (the primary weapon system) and artillery (the supporting weapon system). The problem required the determination of the optimal strategy for each side in distributing the fire of its supporting

weapon system over enemy target types. This according to the criterion of the force ratio of infantry at the end of the prescribed duration attack scenario. Kawara concludes that each side's optimal strategy for the distribution of its supporting weapon system's fire is to always concentrate all fire on the enemy's supporting weapon system (counter-battery fire) during the early stages of battle (if the total prescribed length of battle is sufficient) and at an appropriate time switch to concentration of all fires on the enemy's infantry.

Kawara concludes that this switching time is independent of current strength of either side, and dependent only on the effectiveness of both sides' supporting units. Moreover, an optimal strategy has the property of always requiring concentration of supporting fires on enemy infantry during the final stages of battle. Essentially, Kawara concludes that optimal strategies are independent of force levels. This conclusion is only true, however, if the appropriate side's supporting weapon system is not reduced to a zero force level before a critical time as illustrated by Isaacs [4]. Given that this is the case, Taylor shows in [13] that the optimal strategies are a function of the criterion functional for a given set of combat attrition equations, and that Kawara considered the only type of criterion functional (objective function) which yields optimal strategies independent of force levels.

The purpose of this work is to illustrate the dependence of the structure of the optimal strategies, for the case in which neither sides' supporting units are all lost, upon model form. What is considered here is a slight variation in Kawara's problem for which the structure of the optimal strategy of one of the combatants is significantly different from that in the original problem [5]: the optimal strategy of one combatant depends directly upon the enemy's force levels and is no longer to always concentrate all fire on either the enemy's primary or supporting weapons system. The optimal strategy of one side to sometimes split its fire is very similar to that which occurs in a one-sided (optimal control) problem considered by Taylor [12] for the optimal distribution of fire by a homogeneous force in combat against homogeneous forces. [12] was, in fact, partial motivation for the examination here of another attrition structure in Kawara's problem.

II. THE MODEL

A. THE DIFFERENTIAL GAME FORMULATION

Consider the following model, cast in the form of a differential game, of a prescribed duration battle describing combat between two opposing forces each with primary and supporting elements.

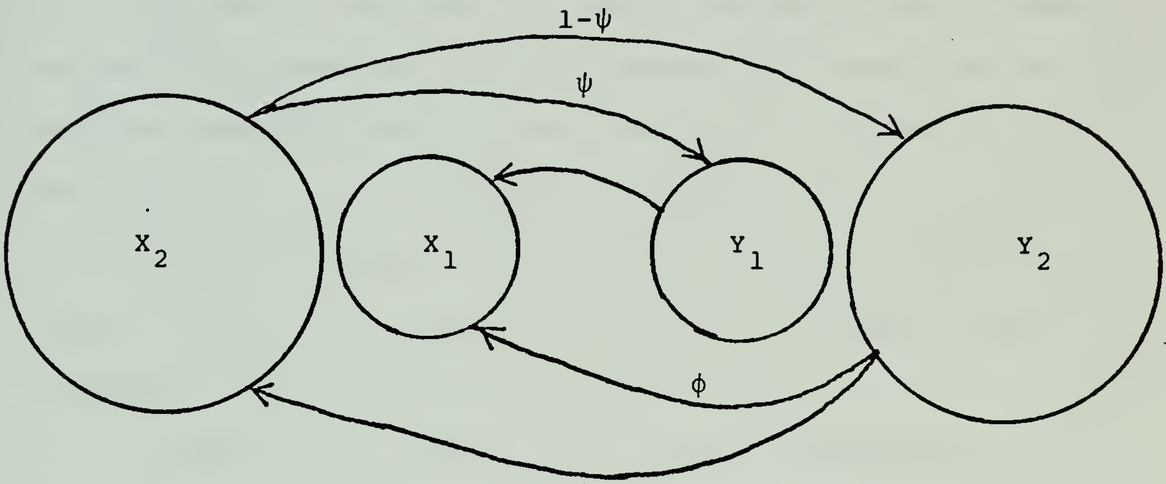

FIGURE 1

$$\underset{\Psi}{\text{Maximize}}\ \underset{\Phi}{\text{Minimize}}\ J = X_1(T)/Y_1(T), \text{ with } T \text{ specified} \tag{1}$$

subject to:

$$\frac{dX_1}{dt} = -a_1X_1Y_1 - \phi a_2X_1Y_2 \tag{2}$$

$$\frac{dX_2}{dt} = -(1-\phi)cY_2 \tag{3}$$

$$\frac{dY_1}{dt} = -\psi b_1Y_1X_2 \tag{4}$$

$$\frac{dY_2}{dt} = -(1-\psi)b_2X_2Y_2 \qquad (5)$$

with initial conditions

$$X_1(t=0) = X_1^\circ \qquad Y_1(t=0) = Y_1^\circ$$
$$X_2(t=0) = X_2^\circ \qquad Y_2(t=0) = Y_2^\circ$$

and $\qquad X_1, X_2, Y_1, Y_2 \geq 0 \;, \quad 0 \leq \phi,\ \psi \leq 1$

where all symbols are defined in Table I.

This model represents a modification of that presented in Ref. [5] inasmuch as the differential equations describing the combat dynamics have been altered to allow Y's primary force, Y_1, to attrite X's primary force, X_1. Also, the Lanchester square law attrition of X's supporting force, X_2, on Y's supporting force, Y_2 has been modified to linear law attrition.

A scenario that leads to the above mathematical formulation is given below in order that insights gained from the mathematical solution to this problem can take on other than abstract significance.

B. COMBAT SCENARIO

A battlefield scenario to which the above model might be applied involves a conflict between two forces, X and Y, each with two types of elements; infantry units, X_1 and Y_1, and supporting artillery units, X_2 and Y_2. Consider the situation in which the Y infantry forces are

tasked with defending a fortified position and are supported in this effort by Y's supporting artillery, which have the capability of providing a mixture of offensive fire on attacking units and counter battery fire. The mission of X's infantry, X_1, on the other hand, is to attack that defended position, supported by X's artillery, X_2, which can fire a mixture of support for X_1 in the attack, or provide counter battery fire on Y_2. Since the X forces are in the attack, moving toward Y_1, they are incapable of causing any attrition on either of Y's elements. Y_1, however, although not within its own weapons range of X_2 can inflict casualties upon the attacking infantry units, X_1. In this instance the combat duration T corresponds to the time required for X's infantry to close to its final assault position where it could begin inflicting casualties on Y_1. In this scenario X and Y are faced with allocating supporting artillery resources, over the duration of the battle, so that the optimal value of the ratio of remaining X infantry to remaining Y infantry at time T is reached.

The scenario presented is the same basically as is presented by Kawara [5] with the addition of the ability of Y's infantry to attrite X's infantry. Taylor [13] shows that an optimal strategy in which a side divides the fire of its supporting weapons system between the enemy's primary (infantry) and supporting systems <u>can</u>

only occur when the enemy's infantry has some effectiveness against his infantry. Is is this division of fires, or singular control in the control theory sense, which most dramatically illustrates the force level dependence of optimal strategies in this problem.

III. DEVELOPMENT OF NECESSARY CONDITIONS OF OPTIMALITY

The development of the solution to the differential game presented in Chapter Two is similar to that for an optimal control problem. However, its complete solution requires more than the well known MAX-MIN principle [4]: the theory of singular extremals must be utilized in its solution. A brief discussion of that theory is given in Ref. [12].

Before applying the above theory to the problem at hand it should be pointed out that the combat dynamics, i.e. Equations (1) through (5), and the fact that the combat duration T is finite, yield that X_1, Y_1, Y_2 will always be greater than zero. It is possible that X_2 could be forced to zero; however, since X_2 controls the decision variable ψ, and the case of interest is that in which no supporting units are forced to a zero level, X_2 will be assumed positive.

The solution procedure begins by forming the Hamiltonian

$$H(X_1,X_2,Y_1,Y_2,p_1,p_2,q_1,q_2,\psi,\phi,t)=\{(-a_1X_1Y_1-\phi a_2X_1Y_2)p_1(t)$$
$$-(1-\phi)cY_2p_2(t)-\psi b_1Y_1X_2q_1(t)\ -(1-\psi)b_2X_2Y_2q_2(t)\}$$

where $p_i(t)$ and $q_i(t)$ are dual variables associated with X_i and Y_i, respectively.

By the MAX-MIN principle

$$\underset{\psi}{\text{MAX}}\ \underset{\phi}{\text{MIN}}\ H(X_1,X_2,Y_1,Y_2,p_1,p_2,q_1,q_2,\psi,\phi,t) = \underset{\psi}{\text{MAX}}\ \underset{\phi}{\text{MIN}}\ J$$

However, because of separability properties of the Hamiltonian, this is equivalent to

$$\min_{\phi}\ \{-\phi a_2X_1Y_2p_1 + \phi cY_2p_2\}$$
$$\max_{\psi}\ \{-\psi b_1Y_1X_2q_1 + \psi b_2X_2Y_2q_2\}$$

or

$$\min_{\phi}\ \{-\phi Y_2(a_2X_1p_1-cp_2)\}$$
$$\max_{\psi}\ \{-\psi X_2(b_1Y_1q_1-b_2Y_2q_2)\}$$

Defining $S_\psi(t)$ and $S_\phi(t)$ as the switching functions for X and Y, respectively, where

$$S_\psi(t) = b_1Y_1q_1 - b_2Y_2q_2 \tag{7}$$
$$S_\phi(t) = a_2X_1p_1 - cp_2 \tag{8}$$

it is clear that the extremal controls are given by

$$\psi^*(t) = \begin{cases} 1 & \text{if } S_\psi(t) < 0 \\ 0 & \text{if } S_\psi(t) > 0 \\ \text{undetermined} & \text{if } S_\psi(t) = 0 \end{cases} \tag{9}$$

and

$$\phi^*(t) = \begin{cases} 1 & \text{if } S_\phi(t) > 0 \\ 0 & \text{if } S_\phi(t) < 0 \\ \text{undetermined} & \text{if } S_\phi(t) = 0 \end{cases} \tag{10}$$

The time history of using these controls is given by the state and adjoint equations below with boundary conditions on dual variables as derived from the MAX-MIN principle.

State Equations

$$\frac{dX_1}{dt} = -a_1X_1Y_1-\phi a_2X_1Y_2 \quad ; \quad X_1(t=0) = X_1^\circ \tag{11}$$

$$\frac{dX_2}{dt} = -(1-\phi)cY_2 \quad ; \quad X_2(t=0) = X_2^\circ \tag{12}$$

$$\frac{dY_1}{dt} = -\psi b_1Y_1X_2 \quad ; \quad Y_1(t=0) = Y_1^\circ \tag{13}$$

$$\frac{dY_2}{dt} = -(1-\psi)b_2X_2Y_2 \quad ; \quad Y_2(t=0) = Y_2^\circ \tag{14}$$

Adjoint Equations

$$\frac{dp_1}{dt} = \frac{-\partial H}{\partial X_1} = a_1Y_1p_1+\phi a_2Y_2p_2 \quad ; \quad p_1(t=T)=\frac{1}{Y_1(T)} \tag{15}$$

$$\frac{dp_2}{dt} = \frac{-\partial H}{\partial X_2} = \psi b_1Y_1q_1+(1-\psi)b_2Y_2q_2; \quad p_2(t=T)= 0 \tag{16}$$

$$\frac{dq_1}{dt} = \frac{-\partial H}{\partial Y_1} = a_1X_1p_1+\psi b_1X_2q_1 \quad ; \quad q_1(t=T)=-\frac{X_1(T)}{Y_1(T)} \tag{17}$$

$$\frac{dq_2}{dt} = \frac{-\partial H}{\partial Y_2} = \phi a_2X_1p_1+(1-\phi)cp_2+(1-\psi)b_2X_2q_2; \quad q_2(t=T)=0 \tag{18}$$

The necessary conditions for extremal controls given in (9) and (10) allow the determination of optimal controls at battle termination T. This is accomplished by noting that

$$S_\psi(t=T) = b_1Y_1(T)q_1(T) -b_2Y_2(T)q_2(T)$$

and

$$S_{\phi}(t=T) = a_2 X_1(T) p_1(T) - cp_2(T)$$

Applying the boundary conditions to Equations (15) through (18), and noting that force levels are all positive for $t \leq T$, yields $S_{\psi}(t=T)<0$ and $S_{\phi}(t=T)>0$ implying that $\psi^*(T)=1$ and $\phi^*(T)=1$.

This information allows the determination of extremal controls when either of the switching functions, $S_{\psi}(t)$ or $S_{\phi}(t)$ are identically zero. It is at this point in the development of the solution that the theory of singular solutions must be called upon.

In the discussion to follow a singular subarc will denote that part of an optimal trajectory on which the MAX-MIN principle cannot be used to determine the control, because the coefficient of the control variable in the Hamiltonian is zero. The term "singular solution" will be used to denote any optimal trajectory which contains one or more singular subarcs.

In partial elaboration relative to the problem at hand, since the Hamiltonian H is a linear function of both control variables, ψ and ϕ, if $\frac{\partial H}{\partial \phi}$ or $\frac{\partial H}{\partial \psi}$ (or equivalently $S_{\psi}(t)$, $S_{\phi}(t)$) vanish for a finite interval of time, then as in the case in (9) and (10) above, all feasible values of the respective control optimize the Hamiltonian. When this occurs the singular control is determined by requiring that the appropriate switching function, and all of

its time derivatives, be zero for that finite time period.

Investigating the behavior of Y's decision variable, where

$$S_{\phi}(t) = a_2 X_1 p_1 - c p_2 = 0$$

and

$$\frac{dS_{\phi}(t)}{dt} = -c\ \{\psi b_1 Y_1 q_1 + (1-\psi) b_2 Y_2 q_2\}$$

The existence of a singular subarc in ϕ requires that $S_{\phi}(t)=0$ for a finite interval of time, implying that $\frac{dS_{\psi}}{dt} = 0.$

But,

$$0 \le \psi \le 1$$

$$Y_1, Y_2 > 0$$

and $q_i = \frac{\partial J}{\partial Y_i}$, where J is the criterion functional. It can be shown that $q_i < 0\ \forall t \in [0,T]$; therefore, $\frac{dS_{\phi}}{dt}$ is represented by a convex combination of negative quantities and cannot be zero. No singular solutions exist in ϕ.

Noting additionally that $\frac{dS_{\phi}}{dt}$ can be rewritten such that

$$\frac{dS_{\phi}}{dt} = -c\{\psi S_{\psi}(t) + b_2 Y_2 q_2\}$$

and the product $\psi S_{\psi}(t) \le 0 \forall t \in [0,T]$ implying that

$\frac{dS_\phi}{dt} > 0 \; \forall t \in [0,T]$, but $S_\phi(T) > 0$ so there is at most one switch in Y's decision variable from $\phi^*=0$ to $\phi^*=1$ which would occur when $S_\phi(t) = 0$.

This information allows for the investigation into singular behavior of X's strategic variable, ψ, under the two possible values of ϕ^*.

Consider first what general conditions are necessary in order for a singular solution to exist in ψ by setting successive time derivatives of $S_\psi(t)$ equal to zero until ψ appears explicitly in one of those derivatives. From

$$S_\psi(t) = b_1Y_1q_1 - b_2Y_2q_2 = 0$$

and

$$\frac{dS_\psi(t)}{dt} = a_1b_1X_1Y_1p_1 - b_2cY_2p_2 - \phi b_2Y_2(a_2X_1p_1 - cp_2) = 0$$

two conditions necessary for the existence of a singular subarc are:

$$b_1Y_1q_1 = b_2Y_2q_2$$

$$a_1b_1X_1Y_1p_1 = b_2Y_2\{\phi a_2X_1p_1 + (1-\phi)cp_2\}$$

Additionally, $\frac{d}{dt}\left(\frac{dS_\psi(t)}{dt}\right)$ contains ψ explicitly, so utilizing the above results the singular control for X is

$$\psi^*(t) = \left(\frac{b_2}{b_1+b_2}\right)\left\{1 - \frac{(1-\phi)cY_2q_2}{X_2[\phi a_2X_1p_1 + (1-\phi)cp_2]}\right\}$$

This expression, however, gives the value of the singular control for X as a function of Y's strategic

variable, ϕ^*. Therefore, there are two singular control values to be considered corresponding to the two possible values of ϕ^*.

1. Case I: ($\phi^*=1$), for this case the conditions necessary for the control to exist are

$$
\begin{aligned}
b_1 q_1 Y_1 &= b_2 q_2 Y_2 \\
a_1 b_1 Y_1 &= a_2 b_2 Y_2
\end{aligned}
\tag{19}
$$

and the singular control is

$$
\psi^*(t) = \frac{b_2}{b_1+b_2} \tag{20}
$$

2. Case II: ($\phi^*=0$), for this case the conditions for the existence of the singular control are

$$
\begin{aligned}
b_1 Y_1 q_1 &= b_2 Y_2 q_2 \\
a_1 b_1 X_1 Y_1 p_1 &= b_2 c Y_2 q_2
\end{aligned}
\tag{21}
$$

and the singular control is

$$
\psi^*(t) = \left(\frac{b_2}{b_1+b_2}\right) \left\{1 - \frac{Y_2 q_2}{X_2 p_2}\right\} \tag{22}
$$

A further check can be made on these singular controls to insure a maximum return from the criterion functional, J, in the form of a further necessary condition, the generalized Legendre-Clebsh condition,

$$
(-1)^k \frac{\partial}{\partial \psi} \left\{\frac{d^{2k}}{dt^{2k}} \left(\frac{\partial H}{\partial \psi}\right)\right\} \leq 0
$$

which necessarily must be met for a singular subarc to yield a maximum return. In the problem at hand it suffices to check this condition for K=1, and, it can be shown that the condition is met for both singular controls.

The external controls as further defined by the above are given as:

$$\phi^*(t) = \begin{cases} 1 & \text{if} \quad S_\phi(t) > 0 \\ 0 & \text{if} \quad S_\phi(t) < 0 \end{cases} \qquad (23)$$

$$\psi^*(t) = \begin{cases} 1 & \text{if} \quad S_\psi(t) < 0 \\ 0 & \text{if} \quad S_\psi(t) > 0 \\ \dfrac{b_2}{b_1+b_2} & \text{if } S_\psi(t)=0 \text{ for a finite period of time and } \phi^*=1 \\ \dfrac{b_2}{b_1+b_2}\left(1-\dfrac{Y_2 q_2}{X_2 p_2}\right) & \text{if } S_\psi(t)=0 \text{ for a finite period of time and } \phi^*=0 \end{cases} \qquad (24)$$

IV. SYNTHESIS OF EXTREMAL CONTROLS

A. METHOD OF SYNTHESIS

By the synthesis of extremal controls is meant the explicit determination of the time history of the extremal controls from initial to terminal state. This is accomplished by combining the extremal controls with integration of the state and adjoint systems of equations. The specific procedure used in the solution to the problem at hand is to begin at battle termination, T, where extremal controls are known, assuming final values for state variables. This in turn yields the boundary conditions on the set of adjoint equations in terms of those final values. The procedure is then to solve the system of differential equations represented by the state and adjoint equations combined with $\frac{dS_\psi}{dt}$ and $\frac{dS_\phi}{dt}$, the differential equations describing the behavior of the switching functions over time. The solution of this system of equations and in particular the values of $S_\psi(t)$ and $S_\phi(t)$ yield the extremal controls over time through Equations (23) and (24).

In order to facilitate this procedure it is convenient to make a change of variable to transform the set of boundary conditions at t=T into initial conditions for the described system of differential equations. This is accomplished by converting to "backwards" time, $\tau=T-t$, and noting that $d\tau = -dt$.

The procedure is then to integrate the above set of differential equations in "backwards" time from $\tau=0$ until the first change in tactics occurs for either X or Y. The time at which that switch occurs is then regarded as an artificial terminal surface from which initial conditions can again be computed for the new system of differential equations as altered by the new control. This procedure is repeated at each succeeding tactic change until $\tau=T$. By beginning at $\tau=0$ with all possible elements of the set of terminal values of the state variables, a field of extremal trajectories, resulting from extremal controls, could be mapped out such that for any initial state in forward time t at least one candidate optimal trajectory could be found that would lead to the set of possible final values and thus max-min of the criterion functional. If more than one trajectory satisfying the derived necessary conditions arrive at $t=T$ at different values of the criterion functional the higher value would be taken and the trajectories leading to other terminal states would be deleted.

Since the control displaying the most interesting behavior is ψ^*, X's decision variable, and since ϕ^*, Y's decision variable, takes on only two values, 0 or 1, it is convenient to break the synthesis of extremal controls into two regions as defined below.

1. Region I:

$\phi^*=1$, $\tau \in [\tau_\psi, 0]$ - or from $t=T$ to the artificial terminal surface described by switch in Y's control, .

2. Region II:

$\phi^*=0$, $\tau \in [T, \tau_\phi]$ - or from $t=t_\phi$ to $t=0$.

The procedure is then to develop the time history of ψ^* over each region.

B. SYNTHESIS OF EXTREMAL CONTROLS - REGION I

The investigation into the behavior of extremal controls proceeds at this point over Region I, where $\phi^*=1$, beginning at $\tau=0$ where the optimal controls at battle termination were previously shown to be $\psi^*=1$, $\phi^*=1$. Looking in "backwards" time from $\tau=0$, there are two contingencies to consider. A strategy change can occur for X, in which case a new artificial terminal surface will be considered, or the strategy change can occur for Y in which case the solution enters Region II.

The system of differential equations transformed to "backwards" time which describe the battle trajectory under controls $\psi^*=1$, $\phi^*=1$ and their initial conditions at $\tau=0$ are given below.

State Equations

$$\frac{dX_1}{d\tau} = X_1(a_1Y_1+a_2Y_2); \quad X_1(\tau=0) = X_1^f \tag{25}$$

$$\frac{dX_2}{d\tau} = 0 \quad ; \quad X_2(\tau=0) = X_2^f \tag{26}$$

$$\frac{dY_1}{d\tau} = b_1Y_1X_2 \qquad ; \; Y_1(\tau=0) = Y_1^f \tag{27}$$

$$\frac{dY_2}{d\tau} = 0 \qquad ; \; Y_2(\tau=0) = Y_2^f \tag{28}$$

Adjoint Equations

$$\frac{dp_1}{d\tau} = -p_1(a_1Y_1+a_2Y_2) \qquad ; \; p_1(\tau=0) = \frac{1}{Y_1^f} \tag{29}$$

$$\frac{dp_2}{d\tau} = -b_1Y_1q_1 \qquad ; \; p_2(\tau=0) = 0 \tag{30}$$

$$\frac{dq_1}{d\tau} = -a_1X_1p_1-b_1X_2q_1 \qquad ; \; q_1(\tau=0) = -\frac{X_1^f}{(Y_1^f)^2} \tag{31}$$

$$\frac{dq_2}{d\tau} = -a_2X_1p_1 \qquad ; \; q_2(\tau=0) = 0 \tag{32}$$

Switching Functions

$$\frac{dS_\phi}{d\tau} = b_1cY_1q_1 \qquad ; \; S_\phi(\tau=0) = \frac{a_2X_1^f}{Y_1^f} \tag{33}$$

$$\frac{dS_\psi}{d\tau} = -X_1p_1(a_1b_1Y_1-a_2b_2Y_2) ; \; S_\psi(\tau=0) = -\frac{b_1X_1^f}{Y_1^f} \tag{34}$$

The following solutions to the above system of equations were developed.

State Variables

$$X_1(\tau) = X_1^f \; \text{EXP}[a_2Y_2^f\tau - \frac{a_1Y_1^f}{b_1X_2^f}(1-e^{b_1X_2^f\tau})] \tag{35}$$

$$X_2(\tau) = X_2^f \tag{36}$$

$$Y_1(\tau) = Y_1^f \; e^{b_1X_2^f\tau} \tag{37}$$

$$Y_2(\tau) = Y_2^f \tag{38}$$

Adjoint Variables

$$p_1(\tau) = \frac{1}{Y_1^f} \text{EXP} \left[-a_2 Y_2^f \tau + \frac{a_1 Y_1^f}{b_1 X_2^f} (1-e^{b_1 X_2^f \tau})\right] \tag{39}$$

$$p_2(\tau) = \{(\frac{b_1 X_1^f X_2^f - a_1 Y_1^f X_1^f}{X_2^f Y_1^f})\tau - \frac{a_1 X_1^f}{b_1 (X_2^f)^2}(1-e^{b_1 X_2^f \tau})]\} \tag{40}$$

$$q_1(\tau) = -\frac{X_1^f}{Y_1^f} \{(\frac{b_1 X_2^f - a_1 Y_1^f}{b_1 X_2^f Y_1^f}) e^{-b_1 X_2^f \tau} + \frac{a_1}{b_1 X_2^f}\} \tag{41}$$

$$q_2(\tau) = -a_2 \frac{X_1^f}{Y_1^f} \tau \tag{42}$$

Switching Functions

$$S_\phi(\tau) = -c\{(\frac{b_1 X_1^f X_2^f - a_1 Y_1^f X_1^f}{X_1^f Y_1^f})\tau - \frac{a_1 X_1^f}{b_1 (X_2^f)^2}(1-e^{b_1 X_2^f \tau}) + \frac{a_2 X_1^f}{Y_1^f}\} \tag{43}$$

$$S_\psi(\tau) = \frac{X_1^f}{Y_1^f} \{\frac{a_1 Y_1^f}{X_2^f}(1-e^{b_1 X_2^f \tau}) + a_2 b_2 Y_2^f \tau - b_1\} \tag{44}$$

Utilizing Equations (43) and (44), the time of the first tactic change can be determined as the time at which either $S_\psi(\tau)$ or $S_\phi(\tau)$ first goes to zero. From this point it is obvious that if $S_\phi(\tau) = 0$ but $S_\psi(\tau) < 0$ then the tactic change that occurs is $\phi^*=1$ to $\phi^*=0$ with $\tau = \tau_\phi$, and

the solution is now in Region II. If, however, the final values of the state variables chosen were such that $S_\psi(\tau)=0$, and $S_\phi(\tau)$ remains greater than zero, then the switch that occurs is in X's decision variable, ψ^*, from $\psi^*=1$ to $\psi^*=0$ or to $\psi^* = \frac{b_2}{b_1+b_2}$, the singular control. This describes three cases to be examined based on the decision variable, ψ^*, in Region I.

1. Case I

   ψ^* switches from 1 to $\frac{b_2}{b_1+b_2}$, the singular control.

   a. Sub-case I(a)

      $\psi^*=1$ to $\psi^*= \frac{b_2}{b_1+b_2}$ and ψ^* remains the singular subarc until $\tau = \tau_\phi$

   b. Sub-case I(b)

      $\psi^*=1$ to $\psi^*= \frac{b_2}{b_1+b_2}$ to $\psi^*=1$ before $\tau = \tau_\phi$

   c. Sub-case I(c)

      $\psi^*=1$ to $\psi^*= \frac{b_2}{b_1+b_2}$ to $\psi^*=0$ before $\tau = \tau_\phi$

2. Case II

   $\psi^*=1$ to $\psi^*=0$ until $\tau = \tau_\phi$

3. Case III

   $\psi^*=1$ $\forall \tau \in [\tau_\phi, 0]$ corresponding to no switch in ψ^* before $\phi^*=1$ to $\phi^*=0$.

It is convenient at this point to pursue the development of Case I, and in particular since the behavior of extremal controls is being examined in "backwards" time as a function of value of state variables at battle termination, to determine what relationships between those

final values satisfy the previously derived necessary conditions for attaining the singular subarc in $\tau \in [\tau_\phi, 0]$. Those conditions are

$$b_1 Y_1 q_1 = b_2 Y_2 q_2$$

$$a_1 b_1 Y_1 = a_2 b_2 Y_2$$

at $\tau_\psi{}^*$, where $\tau_\psi{}^*$ is the "backwards' time of entry onto the singular subarc where the singular control is $\psi^* = \dfrac{b_2}{b_1 + b_2}$.

Forming the ratio of the above conditions yields the below single condition necessary for the original conditions and for obtaining the singular subarc:

$$a_1 q_2 = a_2 q_1 \tag{45}$$

From (45) above and Equations (41) and (42) the following transcendental equation for $\tau_\psi{}^*$ results.

$$\tau_\psi{}^* = \frac{1}{b_1 X_2{}^f} + \left(\frac{1}{a_1 Y_1{}^f} - \frac{1}{b_1 X_2{}^f}\right) e^{-b_1 X_2{}^f \tau_\psi{}^*} \tag{46}$$

Using the second of the conditions given in (19)

$$a_1 b_1 Y_1 = a_2 b_2 Y_2 \quad \text{at } \tau_\psi{}^*$$

and Equations (37) and (38) evaluated at $\tau_\psi{}^*$ yield the following relationship between $Y_1{}^f$ and $Y_2{}^f$ which satisfies the conditions required to have a singular solution in Region I.

$$Y_2^f = \frac{a_1 b_1}{a_2 b_2} Y_1^f e^{b_1 X_2^f \tau_\psi^*} \tag{47}$$

Since (46) and (47) above yield the values of τ_ψ^* and Y_2^f from given final values of the other state variables, X_1^f, X_2^f, Y_1^f, it is convenient to base further investigation on the ratio $\frac{Y_1^f}{Y_2^f}$, with $\frac{Y_1^{f*}}{Y_2^{f*}}$ being the particular value of that ratio which insures entry onto the singular subarc where $\psi^* = \frac{b_2}{b_1+b_2}$ at τ_ψ^*.

The convenience of using this ratio lies in the fact that it can be shown that

$$\frac{Y_1^f}{Y_2^f} > \frac{Y_1^{f*}}{Y_2^{f*}} \Longrightarrow \frac{dS_\psi}{d\tau} < 0 \quad \forall \tau \in [0, \tau_\phi]$$

and

$$\frac{Y_1^f}{Y_2^f} < \frac{Y_1^{f*}}{Y_2^{f*}} \Longrightarrow \frac{dS_\psi}{d\tau} > 0 \quad \forall \tau \in [0, \tau_\phi]$$

The three cases presented for examination can now be further defined thusly:

1. <u>Case I</u>:

$\frac{Y_1^f}{Y_2^f} = \frac{Y_1^{f*}}{Y_2^{f*}} \Longrightarrow \psi^*=1$ to $\psi^* = \frac{b_2}{b_1+b_2}$ with subcases a, b, c dependent only on being the singular subarc.

2. <u>Case II</u>:

$\frac{Y_1^f}{Y_2^f} < \frac{Y_1^{f*}}{Y_2^{f*}} \Longrightarrow \psi^*=1$ to $\psi^*=0$, the only tactic change

that occurs in $\tau \in [\tau_\phi, 0]$ since $S_\psi(\tau)$ is monotone in that Region.

3. Case III:

$$\frac{Y_1^f}{Y_2^f} > \frac{Y_1^{f*}}{Y_1^{f*}} \Rightarrow \psi^*=1 \quad \forall \tau \in [\tau_\phi, 0]$$

These cases are developed fully with respect to the behavior of state and adjoint variables and switching times τ_ϕ and τ_ψ in order to obtain a complete synthesis of extremal trajectories in Region I as a function of final state variable values.

1. Case I

$$\frac{Y_1^f}{Y_2^f} = \frac{Y_1^{f*}}{Y_2^{f*}}, \; \psi^*=1 \text{ to } \psi^*= \frac{b_2}{b_1+b_2}, \text{ the singular}$$

control in Region I.

The history of the conflict for $\tau \in [0, \tau_\psi^*]$ is given by Equations (35) through (44). The system of differential equations given below describe the battle history for the period in "backwards" time where $\psi^*= \frac{b_2}{b_1+b_2}$, $\phi^*=1$. τ_ψ^* is treated as an artificial terminal surface so that boundary conditions can be computed by evaluating Equations (35) through (44) at τ_ψ^*.

State Equations

$$\frac{dX_1}{d\tau} = X_1(a_1Y_1 + a_2Y_2) \tag{48}$$

$$\frac{dX_2}{d\tau} = 0 \tag{49}$$

$$\frac{dY_1}{d\tau} = \frac{b_1b_2}{b_1+b_2} Y_1X_2 \tag{50}$$

$$\frac{dY_2}{d\tau} = \frac{b_1b_2}{b_1+b_2} Y_2X_2 \tag{51}$$

Adjoint Equations

$$\frac{dp_1}{d\tau} = -p_1(a_1Y_1+a_2Y_2) \tag{52}$$

$$\frac{dp_2}{d\tau} = -\frac{b_1b_2}{b_1+b_2} (Y_1q_1+Y_2q_2) \tag{53}$$

$$\frac{dq_1}{d\tau} = -a_1X_1p_1 - \frac{b_1b_2}{b_1+b_2} X_2q_1 \tag{54}$$

$$\frac{dq_2}{d\tau} = -a_2X_1p_1 - \frac{b_1b_2}{b_1+b_2} X_2q_2 \tag{55}$$

Switching Functions

$$\frac{dS_\phi}{d\tau} = cb_2Y_2q_2 \tag{56}$$

$$\frac{dS_\psi}{d\tau} = -X_1p_1(a_1b_1Y_1-a_2b_2Y_2) \tag{57}$$

From Equations (48) through (57) the following closed form solutions were developed.

State Equations

$$X_1(\tau) = X_1(\tau_\psi{*})\mathrm{EXP}[-\frac{1}{VX_2{}^f}(a_1Y_1(\tau_\psi{*})+a_2Y_2{}^f) (1 - e^{VX_2{}^f(\tau-\tau_\psi{*})})] \tag{58}$$

$$X_2(\tau) = X_2{}^f \tag{59}$$

$$Y_1(\tau) = Y_1(\tau_\psi^*)e^{VX_2^f(\tau-\tau_\psi^*)} \quad (60)$$

$$Y_2(\tau) = Y_2^f \; e^{VX_2^f(\tau-\tau_\psi^*)} \quad (61)$$

<u>Adjoint Equations</u>

$$p_1(\tau) = p_1(\tau_\psi^*)\mathrm{EXP}[-\frac{1}{VX_2^f}(a_1Y_1(\tau_\psi^*)+a_2Y_2^f)(1-e^{VX_2^f(\tau-\tau_\psi^*)})] \quad (62)$$

$$p_2(\tau) = p_2(\tau_\psi^*)+V\{[Y_1(\tau_\psi^*)(q_1(\tau_\psi^*)+\frac{a_1X_1^f}{VX_2^fY_1^f})+Y_2^f$$

$$(q_2(\tau_\psi^*)+\frac{a_2X_1^f}{VX_2^fY_1^f})](\tau-\tau_\psi^*)+\frac{X_1^f}{(VX_2^f)^2Y_1^f}$$

$$(a_1Y_1(\tau_\psi^*)+a_2Y_2^f)(1-e^{VX_2^f(\tau-\tau_\psi^*)})\} \quad (63)$$

$$q_1(\tau) = [q_1(\tau_\psi^*)+\frac{a_1X_1^f}{VX_2^fY_1^f}]e^{-VX_2^f(\tau-\tau_\psi^*)}-\frac{a_1X_1^f}{VX_2^fY_1^f} \quad (64)$$

$$q_2(\tau) = [q_2(\tau_\psi^*)+\frac{a_2X_1^f}{VX_2^fY_1^f}]e^{-VX_2^f(\tau-\tau_\psi^*)}-\frac{a_2X_1^f}{VX_2^fY_1^f} \quad (65)$$

where $V = \frac{b_1b_2}{b_1+b_2}$.

For case I(a), where $\psi^* = \frac{b_2}{b_1+b_2}$ $\tau\in[\tau_\phi,\tau_\psi^*]$ the switching function which yields τ_ϕ is

$$S_\phi(\tau) = S_\phi(\tau_\psi^*)+\frac{cb_1b_2}{b_1+b_2}\{[Y_1(\tau_\psi^*)(q_1(\tau_\psi^*)+\frac{a_1X_1^f}{VX_2^fY_1^f})$$

$$+Y_2^f(q_2(\tau_\psi^*)+\frac{a_2X_1^f}{VX_2^fY_1^f})](\tau-\tau_\psi^*) +$$

$$\frac{X_1^f}{(VX_2^f)^2 Y_1^f}(a_1Y_1(\tau_\psi^*)+a_2Y_2^f)(1-e^{VX_2^f(\tau-\tau_\psi^*)})\} \tag{66}$$

τ_ϕ is given when $S_\phi(\tau) = 0$.

In considering subcases I(b) and I(c) it must be pointed out that these cases are included to complete the trajectory field in "backwards" time. There are no means of analytically solving for τ_e, the "backwards" time of exit from the singular subarc (corresponding to entry onto the singular subarc in forward time), these times are chosen in order to provide an extremal trajectory corresponding to all initial states at t=0.

Under subcase I(b), $\psi^*=1$ to $\psi^*=\frac{b_2}{b_1+b_2}$ to $\psi^*=1$,

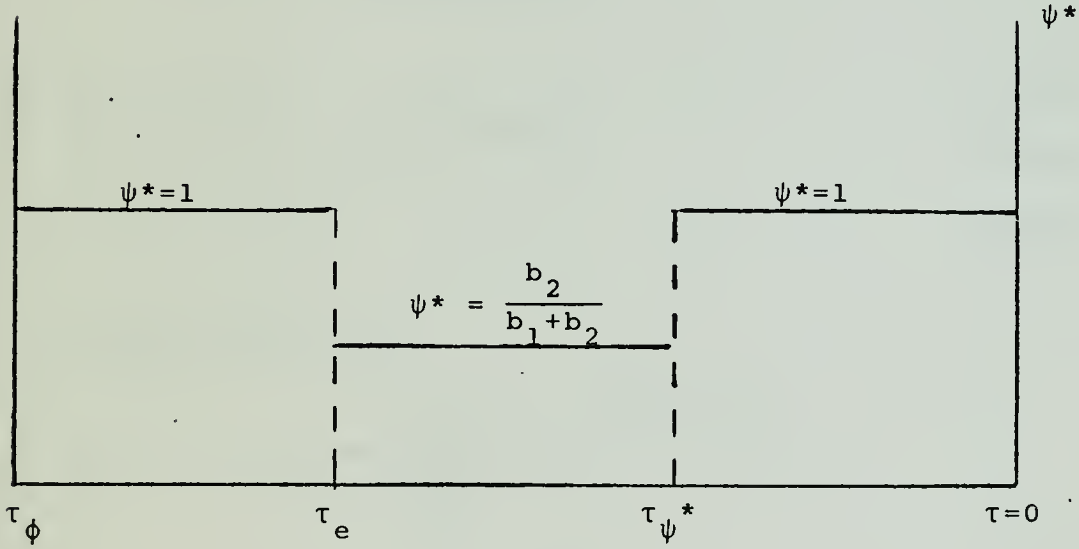

FIGURE 2

the time history of the battle for $\tau \in [0,\tau_\psi^*]$ is again given by Equations (35) through (44). The behavior for

$\tau \in [\tau_{\psi}^*, \tau_e]$ is given by Equations (58) through (66).

The system of differential equations applying for $\tau \in [\tau_e, \tau_{\phi}]$ are Equations (25) through (34), but with initial conditions computed at the artificial terminal surface at τ_e by evaluating Equations (58) through (66) at τ_e.

The behavior of system variables and switching functions for $\tau \in [\tau_e, \tau_{\phi}]$ are given below.

State Equations

$$X_1(\tau) = X_1(\tau_e)\ \mathrm{EXP}[-\frac{a_1Y_1(\tau_e)}{b_1X_2^f}(1-e^{b_1X_2^f(\tau-\tau_e)}) + a_2Y_2(\tau_e)(\tau-\tau_e)] \tag{67}$$

$$X_2(\tau) = X_2^f \tag{68}$$

$$Y_1(\tau) = Y_1(\tau_e)e^{b_1X_2^f(\tau-\tau_e)} \tag{69}$$

$$Y_2(\tau) = Y_2(\tau_e) \tag{70}$$

Adjoint Equations

$$p_1(\tau) = p_1(\tau_e)\ \mathrm{EXP}[\frac{a_1Y_1(\tau_e)}{b_1X_2^f}(1-e^{b_1X_2^f(\tau-\tau_e)}) - a_2Y_2(\tau_e)(\tau-\tau_e)] \tag{71}$$

$$p_2(\tau) = p_2(\tau_e)-b_1Y_1(\tau_e)\{[q_1(\tau_e) + \frac{a_1X_1^f}{b_1X_2^fY_1^f}](\tau-\tau_e)+\frac{a_1X_1^f}{(b_1X_2^f)^2Y_1^f}e^{b_1X_2^f(\tau-\tau_e)}\} \tag{72}$$

$$q_1(\tau) = [q_1(\tau_e) + \frac{a_1X_1^f}{b_1X_2^f Y_1^f}]\, e^{-b_1X_2^f(\tau-\tau_e)} - \frac{a_1X_1^f}{b_1X_2^f Y_1^f} \qquad (73)$$

$$q_2(\tau) = [q_2(\tau_e) + \frac{a_2X_1^f}{b_1X_2^f Y_1^f}]\, e^{-b_1X_2^f(\tau-\tau_e)} - \frac{a_2X_1^f}{b_1X_2^f Y_1^f} \qquad (74)$$

The switching function for ϕ^* which yields τ_ϕ for $S_\phi(\tau) = 0$ is

$$S_\phi(\tau) = S_\phi(\tau_e) + b_1 c Y_1(\tau_e)\, \{(q_1(\tau_e) + \frac{a_1X_1^f}{b_1X_2^f Y_1^f})$$

$$(\tau-\tau_e) + \frac{a_1X_1^f}{(b_1X_2^f)^2 Y_1^f}\,(1 - e^{b_1X_2^f(\tau-\tau_e)})\} \qquad (75)$$

Subcase I(c): $\psi^*=1$ to $\psi^* = \dfrac{b_2}{b_1+b_2}$ to $\psi^*=0$

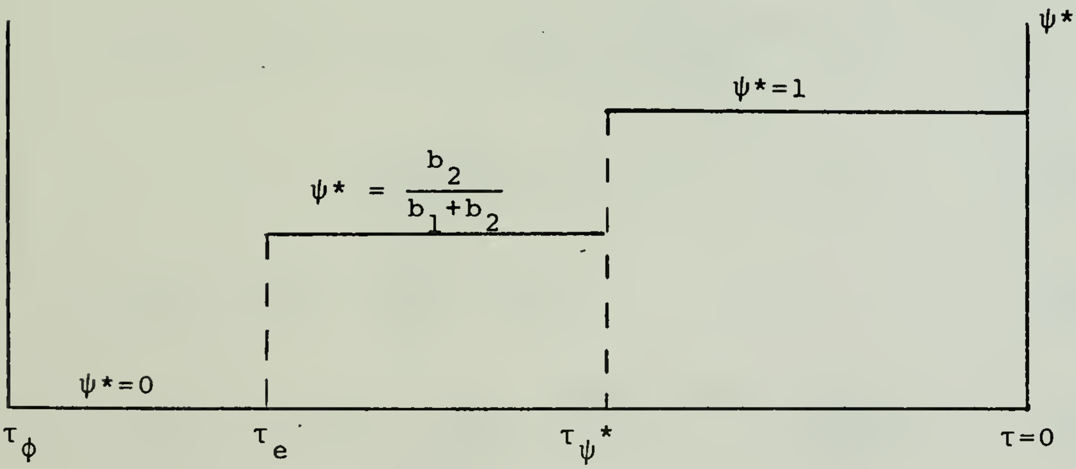

FIGURE 3

Again the time history of the battle for $\tau \in [0, \tau_\psi^*]$ and $\tau \in (\tau_\psi^*, \tau_e]$ are given by Equations (35) through (44) and (58) through (66) respectively.

The solutions to the set of differential equations describing the conflict for $\tau \in (\tau_e, \tau_\phi]$ with extremal controls $\psi^*=0$ and $\phi^*=1$ are given below.

State Equations

$$X_1(\tau) = X_1(\tau_e)\ \mathrm{EXP}[a_1Y_1(\tau_e)(\tau-\tau_e)+ \frac{a_2Y_2(\tau_e)}{b_2X_2^f}\ e^{b_2X_2^f(\tau-\tau_e)}] \quad (76)$$

$$X_2(\tau) = X_2^f \quad (77)$$

$$Y_1(\tau) = Y_1(\tau_e) \quad (78)$$

$$Y_2(\tau) = Y_2(\tau_e)\ e^{b_2X_2^f(\tau-\tau_e)} \quad (79)$$

Adjoint Equations

$$p_1(\tau) = p_1(\tau_e)\ \mathrm{EXP}[-a_1Y_1(\tau_e)(\tau-\tau_e)+ \frac{a_2Y_2(\tau_e)}{b_2X_2^f}\ e^{b_2X_2^f(\tau-\tau_e)}] \quad (80)$$

$$p_2(\tau) = p_2(\tau_e)-b_2Y_2(\tau_e)[(q_2(\tau_e)+ \frac{a_2X_1^f}{b_2X_2^fY_1^f})(\tau-\tau_e) + \frac{a_2X_1^f}{(b_2X_2^f)^2Y_1^f}\ (1 - e^{b_2X_2^f(\tau-\tau_e)})] \quad (81)$$

$$q_1(\tau) = q_1(\tau_e) - \frac{a_1X_1^f}{Y_1^f}\ (\tau-\tau_e) \quad (82)$$

$$q_2(\tau) = [(q_2(\tau_e)+ \frac{a_2X_1^f}{b_2X_2^fY_1^f})\ e^{-b_2X_2^f(\tau-\tau_e)} - \frac{a_2X_1^f}{b_2X_2^fY_1^f}] \quad (83)$$

The switching function for ϕ^*, $S_\phi(\tau)$ which yields τ_ϕ when $S_\phi(\tau) = 0$ is

$$S_\phi(\tau) = S_\phi(\tau_e) + cb_2Y_2(\tau_e)\left[\left(q_2(\tau_e) + \frac{a_2X_1^f}{b_2X_2^fY_1^f}\right)(\tau-\tau_e) + \frac{a_2X_1^f}{(b_2X_2^f)^2Y_1^f}\left(1 - e^{b_2X_2^f(\tau-\tau_e)}\right)\right] \qquad (84)$$

2. Case II:

$$\frac{Y_1^f}{Y_2^f} < \frac{Y_1^{f*}}{Y_2^{f*}} \Rightarrow \psi^*=1 \text{ to } \psi^*=0 \text{ and } \frac{dS_\psi}{d\tau} > 0 \ \forall \tau \in [\tau_\phi, 0].$$

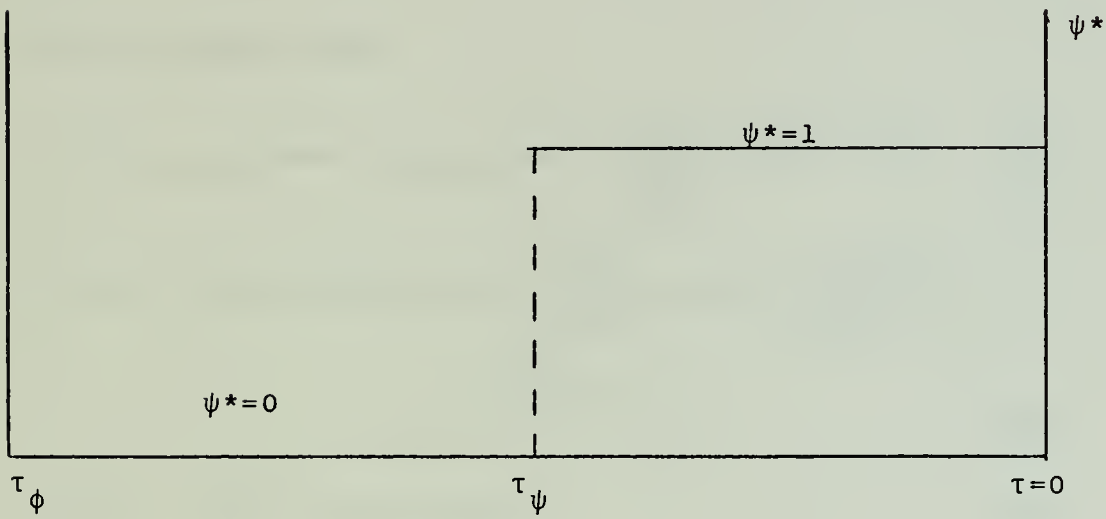

FIGURE 4

The behavior of the conflict for $\tau \in [\tau_\psi, 0]$ is given by Equations (35) through (44). For $\tau \in [\tau_\phi, \tau_\psi)$, where τ_ψ is the time of tactic change for X other than a change to the singular solution, the system of differential equations describing the conflict are identical to those which yield the solutions given for Case I(c). For this case the

initial conditions are derived by computing the values of Equations (35) through (44) at the artificial terminal surface described by τ_ψ. The solution to that set of equations with initial conditions as described are

State Equations

$$X_1(\tau) = X_1(\tau_\psi)\ \mathrm{EXP}[a_1Y_1(\tau_\psi)(\tau-\tau_\psi)+ \frac{a_2Y_2^f}{b_2X_2^f}\, e^{b_2X_2^f(\tau-\tau_\psi)}] \tag{85}$$

$$X_2(\tau) = X_2^f \tag{86}$$

$$Y_1(\tau) = Y_1(\tau_\psi) \tag{87}$$

$$Y_2(\tau) = Y_2^f\ e^{b_2X_2^f(\tau-\tau_\psi)} \tag{88}$$

Adjoint Equations

$$p_1(\tau) = p_1(\tau_\psi)\mathrm{EXP}[-(a_1Y_1(\tau_\psi)(\tau-\tau_\psi)+ \frac{a_2Y_2^f}{b_2X_2^f}\, e^{b_2X_2^f(\tau-\tau_\psi)}] \tag{89}$$

$$p_2(\tau) = p_2(\tau_\psi)-b_2Y_2^f[(q_2(\tau_\psi)+ \frac{a_2X_1^f}{b_2X_2^fY_1^f})(\tau-\tau_\psi)+ \frac{a_2X_1^f}{(b_2X_2^f)^2Y_1^f}(1 - e^{b_2X_2^f(\tau-\tau_\psi)})] \tag{90}$$

$$q_1(\tau) = -\frac{a_1X_1^f}{Y_1^f}(\tau-\tau_\psi)+ q_1(\tau_\psi) \tag{91}$$

$$q_2(\tau) = [(q_2(\tau_\psi)+ \frac{a_2X_1^f}{b_2X_2^fY_1^f})\ e^{-b_2X_2^f(\tau-\tau_\psi)} - \frac{a_2X_1^f}{b_2X_2^fY_1^f}] \tag{92}$$

The switching function which yields τ_ϕ is

$$S_\phi(\tau) = S_\phi(\tau_\psi)+ cb_2Y_2^f[(q_2(\tau_\psi)+ \frac{a_2X_1^f}{b_2X_2^fY_1^f})(\tau-\tau_\psi) + \frac{a_2X_1^f}{(b_2X_2^f)^2Y_1^f}(1 - e^{b_2X_2^f(\tau-\tau_\psi)})] \tag{93}$$

3. Case III:

$$\frac{Y_1^{f}}{Y_2^{f}} > \frac{Y_1^{f*}}{Y_2^{f*}} \Rightarrow \psi^*=1 \quad \forall\tau\in[0,\tau_\phi]$$

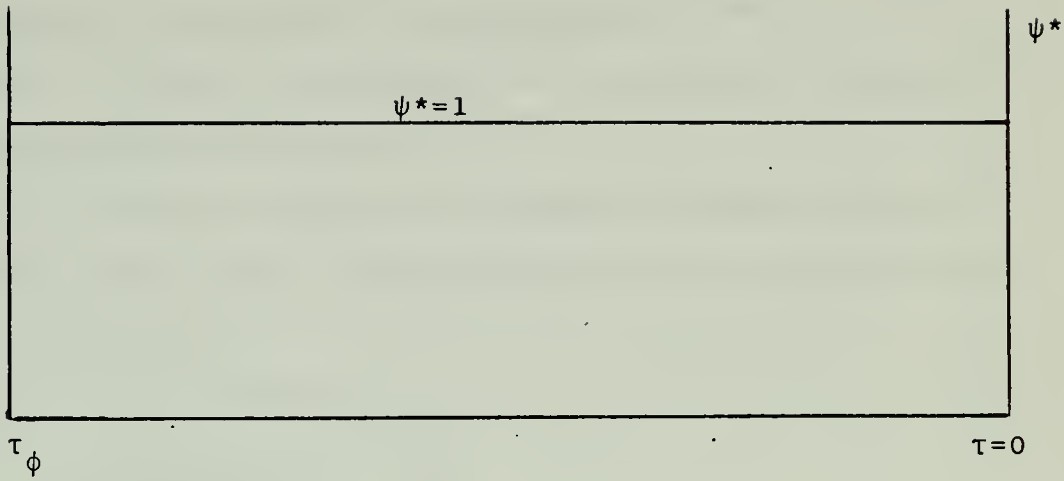

FIGURE 5

The behavior of system variables over time is given by Equations (40) through (44) throughout Region I with the only switch in tactics being from $\phi^*=1$ to $\phi^*=0$. τ_ϕ for this case is given by Equations (44) when $S_\phi(\tau) = 0$.

C. SYNTHESIS OF EXTREMAL CONTROLS - REGION II

The previous discussion has given a complete and somewhat comprehensive description of all possible sequences of values of X's strategic variable meeting the necessary conditions of optimality (extremal trajectories) when $\phi^*=1$. What remains is to describe the remainder of those trajectories over Region II, $\tau\in(\tau_\phi,T]$, where $\phi^*=0$.

Because in general, closed form solutions to the equations describing the battle history over this region are

intractable, the synthesis of extremal trajectories must be accomplished in a less analytical fashion. This will be done by utilizing the experience gained in the behavior of controls over Region I, and insights into the battle history in Region II gleaned from the various systems of state and adjoint equations.

It is again convenient to discuss extremal behavior based on three cases corresponding to the value of ψ^* at τ_ϕ as given below:

1. Case I: $\psi^*=0$ at τ_ϕ
2. Case II: $\psi^*=1$ at τ_ϕ
3. Case III: $\psi^* = \dfrac{b_2}{b_1+b_2}$ at τ_ϕ

Cases I and II can easily be related to those trajectories for which $\dfrac{Y_1^f}{Y_2^f} < \dfrac{Y_1^{f*}}{Y_2^{f*}}$ and $\dfrac{Y_1^f}{Y_2^f} > \dfrac{Y_1^{f*}}{Y_2^{f*}}$ respectively. Also included in these cases are trajectories which exited the singular subarc in Region I and which ended with the proper value of ψ^* at τ_ϕ. For the discussion to follow it is important to recognize that for the portion of extremal trajectories which exited the singular subarc to $\psi^*=0$ or $\psi^*=1$, comments concerning trajectories beginning at $\dfrac{Y_1^f}{Y_2^f} > \dfrac{Y_1^{f*}}{Y_2^{f*}}$ or $\dfrac{Y_1^f}{Y_2^f} < \dfrac{Y_1^{f*}}{Y_2^{f*}}$, respectively, are applicable.

The system of differential equations describing the battle history with $\psi^*=0$, $\phi^*=0$ beginning at the artificial

terminal surface described by τ_ϕ are given below with initial conditions at τ_ϕ.

<u>State Equations</u>

$$\frac{dX_1}{d\tau} = a_1X_1Y_1 \tag{92}$$

$$\frac{dX_2}{d\tau} = cY_2 \tag{93}$$

$$\frac{dY_1}{d\tau} = 0 \tag{94}$$

$$\frac{dY_2}{d\tau} = b_2X_2Y_2 \tag{95}$$

<u>Adjoint Equations</u>

$$\frac{dp_1}{d\tau} = -a_1Y_1p_1 \tag{96}$$

$$\frac{dp_2}{d\tau} = -b_2Y_2q_2 \tag{97}$$

$$\frac{dq_1}{d\tau} = -a_1X_1p_1 \tag{98}$$

$$\frac{dq_2}{d\tau} = -(cp_2+b_2X_2q_2) \tag{99}$$

Switching Functions

$$\frac{dS_\phi}{d\tau} = b_2cY_2q_2 \tag{100}$$

$$\frac{dS_\psi}{d\tau} = b_2cY_2p_2 - a_1b_1X_1Y_1p_1 \tag{101}$$

Recalling that for values of $\dfrac{Y_1^{\,f}}{Y_2^{\,f}} < \dfrac{Y_1^{\,f*}}{Y_2^{\,f*}}$ which correspond to $\psi^*=0$ at τ_ϕ, $\dfrac{dS_\psi}{d\tau} > 0$, $\forall\tau\in[\tau_\phi,0]$, yielding

$$a_1b_1Y_1 < a_2b_2Y_2 \tag{102}$$

additionally (94) and (95) above show that Y_1 remains constant under $\psi^*=0$, $\phi^*=0$ strategies while Y_2 is increasing, implying that (102) will remain true as long as the above controls hold. The monotonacity of $S_\phi(\tau)$ yields,

$$S_\phi(\tau) = a_2X_1p_1 - cp_2 \leq 0 \quad \forall\tau\in[\tau_\phi,T]$$

or $$a_2X_1p_1 \leq cp_2 \tag{103}$$

with equality holding only at τ_ϕ. Equation (101) yields the behavior of $S_\psi(\tau)$ under strategies $\psi^*=0$, $\phi^*=0$ according to

$$\frac{dS_\psi}{d\tau} = b_2Y_2(cp_2) - (a_1b_1Y_1)X_1p_1$$

substituting for $a_1b_1Y_1$ from (102 yields

$$\frac{dS_\psi}{d\tau} = b_2Y_2(cp_2)-(a_1b_1Y_1)X_1p_1 > b_2Y_2(cp_2)-(a_2b_2Y_2)X_1p_1$$

and substituting for cp_2 from (103) yields

$$\frac{dS_\psi}{d\tau} > b_2Y_2(a_2X_1p_1)-(a_2b_2Y_2)X_1p_1 = 0$$

or $\frac{dS_\psi}{d\tau} > 0$ under strategies $\psi^*=0$, $\phi^*=0$. Since $\psi^*=0$ at τ_ϕ then $S_\psi(\tau_\phi) > 0$. Then for Case I there are no further strategy changes for X for $\tau > \tau_\phi$. This can be extended to the fact that once $\psi^*=0$ in Region II it will remain at that value until $\tau = T$.

The battle history of using strategy ψ*=1, ϕ*=0 under Case II are given below:

<u>State Equations</u>

$$\frac{dX_1}{d\tau} = a_1X_1Y_1 \tag{104}$$

$$\frac{dX_2}{d\tau} = cY_2 \tag{105}$$

$$\frac{dY_1}{d\tau} = b_1Y_1X_2 \tag{106}$$

$$\frac{dY_2}{d\tau} = 0 \tag{107}$$

<u>Adjoint Equations</u>

$$\frac{dp_1}{d\tau} = -a_1Y_1p_1 \tag{108}$$

$$\frac{dp_2}{d\tau} = -b_1Y_1q_1 \tag{109}$$

$$\frac{dq_1}{d\tau} = -(a_1X_1p_1+b_1X_2q_2) \tag{110}$$

$$\frac{dq_2}{d\tau} = -cp_2 \tag{111}$$

<u>Switching Functions</u>

$$\frac{dS_\phi}{d\tau} = -cp_2 \tag{112}$$

$$\frac{dS_\psi}{d\tau} = b_2cY_2p_2-a_1b_1X_1Y_1p_1 \tag{113}$$

There are three possible trajectories in Region II under Case II which may occur following strategies which are:

(a) $\psi^*=1$ to $\psi^*=0$

(b) $\psi^*=1 \quad \forall \tau \in [T, \tau_\phi)$

(c) $\psi^*=1$ to $\psi^*= \dfrac{b_2}{b_1+b_2} (1- \dfrac{q_2 Y_2}{X_2 p_2})$

Unfortunately, no general conditions have been found which indicate which of the first two occur, nor has it been determined if the third switch is actually possible. For any given set of parameters numerical techniques can be employed on Equations (104) through (113) using the digital computer to determine which of the first or second possible trajectories actually occur. It would, however, be virtually impossible to determine if the third possibility occurred due to numerical accuracy required to monitor whether or not necessary conditions are met and maintained.

Case III, $\psi^* = \dfrac{b_2}{b_1+b_2}$ at τ_ϕ, again offers three possible trajectories to consider.

(a) $\psi^* = \dfrac{b_2}{b_1+b_2}$ to $\psi^*=0$

(b) $\psi^* = \dfrac{b_2}{b_1+b_2}$ to $\psi^*=1$

(c) $\psi^* = \dfrac{b_2}{b_1+b_2}$ to the singular control in Region II,

$$\psi^* = \frac{b_2}{b_1+b_2} (1 - \frac{Y_2 q_2}{X_2 p_2})$$

It can be shown that the first and second "backwards" time derivatives of $S_\psi(\tau)$, X's switching function, can be written

$$\frac{dS_\psi(\tau)}{d\tau} = b_2Y_2\ S_\phi(\tau) + \frac{X_1^f}{Y_1^f}(a_1b_1Y_1 - a_2b_2Y_2) \tag{114}$$

$$\frac{d^2S_\psi(\tau)}{d\tau^2} = \psi^*b_1X_2\ \frac{dS_\psi}{d\tau} + (b_2Y_2\ \{-\psi^*S_\psi(\tau)$$

$$+ b_2q_2Y_2 + p_2X_2[\psi^*b_1 - (1-\psi^*)b_2]\} \tag{115}$$

Investigating $\frac{dS_\psi}{d\tau}(\tau_\phi)$ it is found, since $S_\phi(\tau_\phi) = 0$ and $a_1b_1Y_1 = a_2b_2Y_2$ at τ_ϕ, assuming continuity of the dual variables, that $\frac{dS_\psi}{d\tau}(\tau_\phi^+) = 0.$ This further implies

$$\frac{d^2S_\psi(\tau_\phi^+)}{d\tau^2} = cb_2Y_2\ \{b_2q_2Y_2+p_2X_2[\psi^*b_1-(1-\psi^*)b_2]\} \tag{116}$$

which, because of the fact that $\frac{dS_\psi}{d\tau}(\tau_\phi^+) = 0$ and $S_\psi(\tau_\phi^+)=0$, controls by its sign which of the three possible extremals is optimal.

Observing that if $\psi^*(\tau)=0$ for $\tau \in (\tau_\phi, \tau_\phi+\delta)$ where $\delta > 0$, then

$$\frac{d^2S_\psi}{d\tau^2}(\tau_\phi^+) = -c(b_2)^2p_2X_2Y_2(1 - \frac{q_2Y_2}{X_2p_2}) < 0$$

so that $\exists\ \delta > 0 \cdot \ni \cdot\ S_\psi(\tau) < 0 \quad \forall \tau \in (\tau_\phi, \tau_\phi+\delta).$

If $\psi^*(\tau)=1$ for $\tau \in (\tau_\phi, \tau_\phi+\delta)$, then

$$\frac{d^2 S_\psi}{d\tau^2}(\tau_\phi^{+}) = cb_2(b_1+b_2)p_2X_2Y_2 \; \{1-(\frac{b_2}{b_1+b_2})(1-\frac{q_2Y_2}{X_2p_2})\}$$

but, $\psi^*(\tau)=1 \Rightarrow S_\psi(\tau) \geq 0$. Also

$$1 > (\frac{b_2}{b_1+b_2})(1 - \frac{q_2Y_2}{X_2p_2})$$

or $b_1p_2X_2 > -b_2q_2Y_2$ implies that $\frac{d^2 S_\psi}{d\tau^2}(\tau_\phi^{+}) > 0 \Rightarrow$

$$\exists \delta > 0 \ni S_\psi(\tau) > 0 \quad \forall \tau \in (\tau_\phi, \tau_\phi+\delta).$$

Also $\psi^*=1$ for $\tau \in (\tau_\phi, \tau_\phi+\delta) \Rightarrow S_\psi(\tau) \geq 0$

for $\tau \in (\tau_\phi, \tau_\phi+\delta) \Rightarrow \frac{d^2 S_\psi(\tau_\phi^{+})}{d\tau^2} \geq 0 \Rightarrow$

$$b_1p_2(\tau_\phi)X_2(\tau_\phi) \geq - b_2q_2(\tau_\phi)Y_2(\tau_\phi).$$

It is also clear that in addition to the necessary conditions for attaining the singular subarc in Region II that

$$\psi^* = (\frac{b_2}{b_1+b_2})(1 - \frac{q_2Y_2}{X_2p_2}) < 1$$

For $\tau \in (\tau_\phi, \tau_\phi+\delta)$, which implies

$$b_1p_2(\tau_\phi)X_2(\tau_\phi) > - b_2q_2(\tau_\phi)Y_2(\tau_\phi)$$

It is true then that for $\tau=\tau_\phi$ and $b_1p_2(\tau_\phi)X_2(\tau_\phi) \geq b_2q_2(\tau_\phi)Y_2(\tau_\phi)$ it is possible to have

$$\psi^* = \begin{cases} 0 \quad \text{if} \quad \dfrac{d^2S_\psi}{d\tau^2}(\tau_\phi^+) > 0 \\ \left(\dfrac{b_2}{b_1+b_2}\right)\left(1-\dfrac{q_2Y_2}{X_2p_2}\right) \text{if} \quad \dfrac{d^2S_\psi}{d\tau^2}(\tau_\phi^+)=0 \\ 1 \quad \text{if} \quad \dfrac{d^2S_\psi}{d\tau^2}(\tau_\phi^+) < 0 \end{cases} \tag{117}$$

If $b_1p_2(\tau_\phi)X_2(\tau_\phi) < -b_2q_2(\tau_\phi)Y_2(\tau_\phi)$ it is only possible to have $\psi^*=0$ for $\tau \in (\tau_\phi, \tau_\phi+\delta)$.

As in Case II general conditions have not been discovered indicating which strategy is followed for all cases but, it is apparent that since the synthesis is proceeding in "backwards" time, there exists some final state at $\tau=0$ for which each of the possible strategies are used at $\tau=\tau_\phi$. This must be the case in order to map an extremal trajectory to every initial state at $\tau=T$.

The battle history for (a) and (b) above are given by Equations (92) through (101) or (104) through (113) respectively. If the singular trajectory is followed the equations below describe battle history.

<u>State Equations</u>

$$\frac{dX_1}{d\tau} = a_1X_1p_1 \tag{118}$$

$$\frac{dX_2}{d\tau} = cY_2 \tag{119}$$

$$\frac{dY_1}{d\tau} = \psi^*b_1Y_1X_2 \tag{120}$$

$$\frac{dY_2}{d\tau} = (1-\psi^*)b_2X_2Y_2 \tag{121}$$

Adjoint Variables

$$\frac{dp_1}{d\tau} = -a_1Y_1p_1 \tag{122}$$

$$\frac{dp_2}{d\tau} = -(\psi^*b_1Y_1q_1+(1-\psi^*)b_2Y_2q_2) \tag{123}$$

$$\frac{dq_1}{d\tau} = -(a_1X_1p_1+\psi^*b_1X_2q_1) \tag{124}$$

$$\frac{dq_2}{d\tau} = -(cp_2+(1-\psi^*)b_2X_2q_2) \tag{125}$$

Switching Function

$$\frac{dS_\phi}{d\tau} = c(\psi^*b_1Y_1q_1+(1-\psi^*)b_2Y_2q_2) \tag{126}$$

$$\frac{dS_\psi}{d\tau} = b_2cY_2p_2-a_1b_1X_1Y_1p_1 \tag{127}$$

Where

$$\psi^* = \left(\frac{b_2}{b_1+b_2}\right)\left(1 - \frac{q_2Y_2}{X_2p_2}\right).$$

V. SOME NUMERICAL EXAMPLES

Using the IBM System 360 Computer, and by following the general procedure for synthesis of extremal controls as outlined in Chapter IV, that is, choosing a final state at $\tau=0$ and integrating the state and adjoint equations in backwards time, numerical solutions to this differential game for several parameter sets were obtained. Each solution is given in the form of a trajectory field (a field of trajectories leading to different final states). Because the state space of the differential game is actually five dimensional, (four state variables and time), and because it has been convenient to base all investigation on the ratio $\frac{Y_1^f}{Y_2^f}$, the trajectory fields are displayed on what is effectively a contour of this five-dimensional space, by plotting the ratio $\frac{Y_1(\tau)}{Y_2(\tau)}$ versus time. This procedure yields a clear picture of the behavior of ψ^*, X's strategy variable, as a function of time. The field of trajectories is generated by varying only Y_1^f.

Integration of the state and adjoint equations in backwards time was accomplished using a fourth order Runge Kutta numerical integration technique. Times at which strategy changes occur for X and Y were computed using a Newton-Rhapson scheme when closed form solutions for the switching functions existed. Where this was not possible

strategy changes were determined to occur with sign changes in the appropriate switching functions. Additionally throughout Region I, where $\phi^*=1$, actual values of system variables were determined from closed form solutions at the above switching times. The Runge Kutta technique employed was verified against one of those closed form solutions and found to produce an error no greater than 10^{-5}.

Two examples of the results obtained are presented here in Figures 6 and 7. The model parameters are given below the figures.

Figure 6 clearly shows the time history of X's strategy for achieving different final states. Also clearly shown are the singular subarcs (the dark heavy lines) which correspond to X splitting his supporting fires between Y_1 and Y_2 (these are called universal surfaces in the terminology of Isaacs [4]. These universal surfaces divide the state space of the game into regions where X's optimal strategy is to concentrate his fire on either Y's supporting elements, $\psi^*=0$ (below the surface), or on his primary force, $\psi^*=1$ (above the surface).

Figure 7 results from a slight modification to the parameter set which yields Figure 6. In this case a different value of Y_2^f is chosen and again Y_1^f is varied. For this particular set of parameters the extremal field is identical to Figure 6 throughout the region where $\phi^*=1$; however, there is a void in the solution space where $\phi^*=0$,

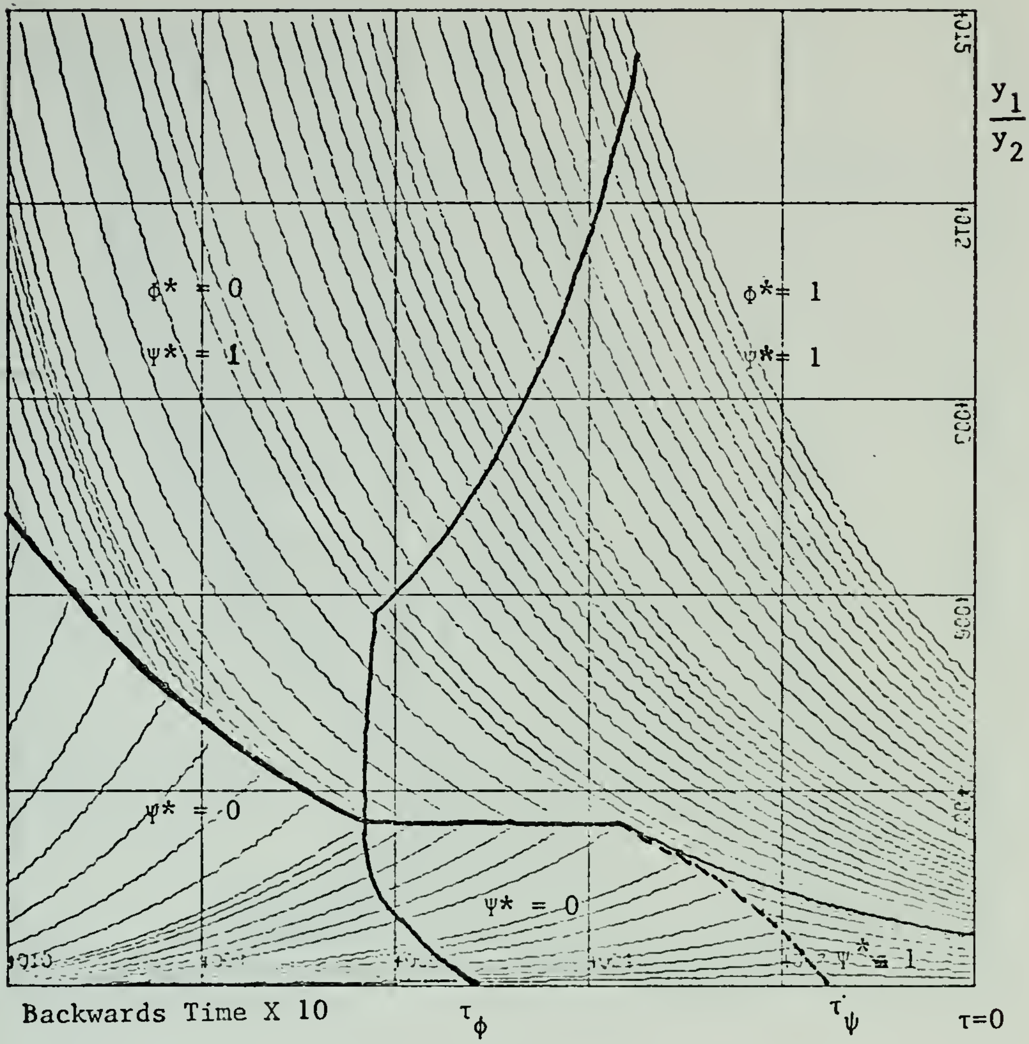

FIGURE 6

$x_1^f = 4.0$

$x_2^f = 8.0$

$y_1^f = 7.0$

$y_2^f = 8.964144$

$a_1 = 0.003$

$a_2 = 0.006$

$b_1 = 0.004$

$b_2 = 0.005$

$c = 0.01$

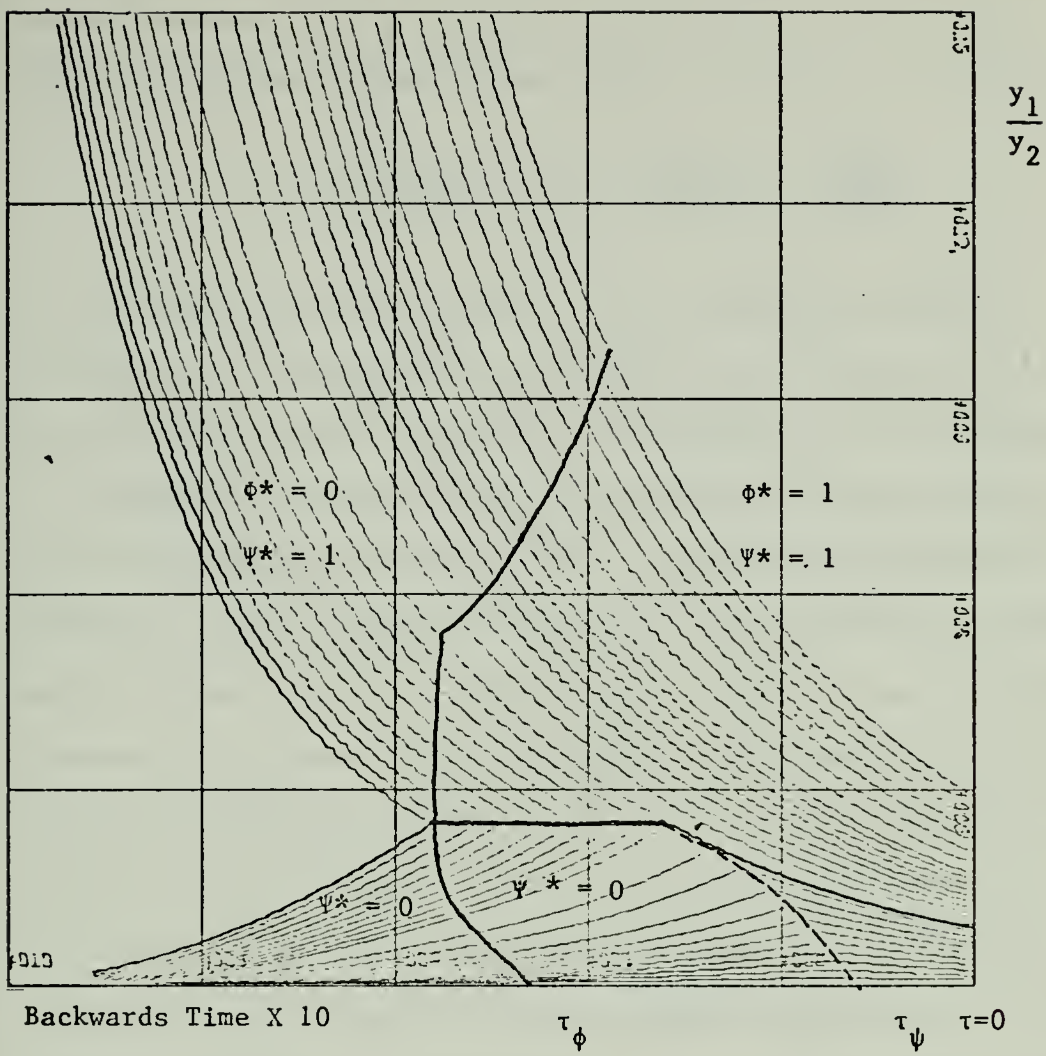

FIGURE 7

$x_1^f = 4.0$ $\qquad a_1 = 0.003$

$x_2^f = 8.0$ $\qquad a_2 = 0.006$

$y_1^f = 10.66666$ $\qquad b_1 = 0.004$

$y_2^f = 11.5967997$ $\qquad b_2 = 0.005$

$c = 0.01$

and the singular subarc appears in Figure 6. This void occurs because at τ_ϕ, where ϕ^* switches from $\phi^*=1$ to $\phi^*=0$ and ψ^* must also change from

$$\psi^* = \frac{b_2}{b_1+b_2} \text{ to } \psi^* = \frac{b_2}{b_1+b_2}\left(1 - \frac{q_2Y_2}{p_2X_2}\right)$$

it is found that the second of the singular controls violates the inequality constraint on ψ^*, that is, $\psi^* > 1$. For the particular case shown in Figure 7 $\psi^* = 1.0543061$.

Extreme care was taken to insure that the violation of the constraint on ψ^* is not a function of numerical accuracy in generating solutions, and it appears at this writing that the dual variables, P_i and q_i, may be discontinuous at τ_ϕ when $\psi^*(\tau_\phi) = \frac{b_2}{b_1+b_2}$. The possibility that this discontinuity occurs is due to the fact that extremal strategies for both X and Y must change simultaneously at the above point. The existence of such discontinuity constitutes a violation of an underlying assumption in the theory utilized in the solution of this differential game. The investigation into that contingency is, however, beyond the scope of this work.

The program which generates the solutions presented here is available from Professor J. G. Taylor, Department of Operations Research, U. S. Naval Postgraduate School.

VI. MILITARY SIGNIFICANCE OF THE MAX-MIN PRINCIPLE

By examination of the switching functions, $S_\phi(t)$ and $S_\psi(t)$, and their derivatives, interpretations of the conditions for when strategy changes occur, and for which strategies are chosen can be derived. The interpretation of these conditions is similar to marginal value interpretation familiar to economists.

It has been previously shown that a change in the Y force's strategy occurs when $S_\phi(t)=0$. This implies that

$$(a_2X_1)p_1 = cp_2 \ .$$

From the Lanchester "linear law" attrition of X's primary force, X_1, and from the "square law" attrition of X's supporting force, X_2, by Y's supporting force, Y_2, the term "a_2X_1" and the coefficient "c" have interpretations "rate of destroying X_1 by an element of Y_2", and "rate of destroying X_2 per unit Y_2", respectively. Additionally, p_i can be interpreted as being the marginal value of each surviving unit of X_i. From this it is easily seen that Y changes his supporting weapon allocation only when Y's rate of destroying the value of X is the same by firing at either X's primary or supporting weapons. Y fires counter-battery, $\phi^*=0$, only so long as $(a_2X_1)p_1 < cp_2$, or as long as the rate of destroying X's value is greater by firing

at his supporting force than by firing at his primary force. Conversely Y fires at X's primary force, $\phi^*=1$, only when he receives the highest rate of destroying X's value with that strategy.

A change in X's strategy occurs only when $S_\psi(t)=0$, or when

$$(b_1Y_1)q_1 = (b_2Y_2)q_2$$

where "b_1Y_1" is the rate of destroying Y_1 by a unit of X_2, and "b_2Y_2" is the rate at which one X_2 attrites Y_2. q_i is the marginal value to the battle of a single Y_i survivor. Again, X changes strategy only when he receives the same rate of destroying Y's value by firing at either Y's supporting force or his primary force. The strategy X chooses when this condition is met depends upon the sign of $\frac{dS_\psi}{dt}$. This derivative has a different form under each of Y's strategies. For $\phi^*=1$

$$\frac{dS_\psi}{dt} = (a_1b_1Y_1 - a_2b_2Y_2)X_1p_1$$

and it has been shown that X splits his fire according to $\psi^* = \frac{b_2}{b_1+b_2}$ if $b_1(a_1Y_1) = b_2(a_2Y_2)$. This condition says that X splits his fires only if he has equal ability to destroy each of the Y elements' kill capability against his primary force. X allocates his fires in such a way as to maintain this condition by allocating his fires in amounts equal to his relative effectiveness against each

of Y's force types. If he does not have equal ability to destroy the kill capability of each of Y's force types against his infantry, he chooses his strategy to achieve that ability. That is, if $b_1(a_1Y_1) < b_2(a_2Y_2)$, and he is less effective in destroying Y_1's capability to destroy his primary force than in destroying Y_2's ability to destroy that force, he concentrates his fire on Y_1, $\psi^*=1$, in an attempt to achieve this parity. The same rational applies in choosing to concentrate on Y_2, $\psi^*=0$.

For $\phi^*=0$, Y firing counter-battery,

$$\frac{dS_\psi}{dt} = b_1(a_1X_1Y_1)p_1 - b_2(cY_2)p_2 \ .$$

For this case it has been shown that if

$$b_1(a_1X_1Y_1)p_1 = b_2(cY_2)p_2$$

then X splits his fires subject to $\psi^* = \frac{b_2}{b_1+b_2}(1 - \frac{q_2Y_2}{p_2X_2})$. The interpretation of this condition is that X splits his fire in order to maintain equal capability of destroying Y's ability to reduce the value of each of his force types. It should be noted, however, that for this split of fire he again allocates in amounts equal to his relative effectiveness against each of Y's force types, but because Y is concentrating fire on X_2, this is weighted by the ratio of total value of surviving Y_2 to total value of surviving X_2. This has the qualitative interpretation of weighting the split by an amount proportional to how much X_2 feels he

must reduce Y_2 in order to account for the attrition he is suffering. This split becomes $\frac{b_2}{b_1+b_2}$ when $\phi^*=1$ and Y_2 is no longer firing at X_2.

As for the case where $\phi^*=1$, X again chooses his strategies so as to attain this parity in his ability to destroy Y's capability of reducing his value through each of his force types.

Figure 6 illustrates this choice of strategies by X which force him to parity in the conditions described above. Once parity in those conditions is achieved, X maintains that parity by splitting his fire between the two Y elements.

VII. CONCLUSIONS

The solution to the differential game presented here illustrates the dependence of optimal strategies on force levels through each opponent's switching function. Where this dependence is not explicit in those switching functions it is implicit via the marginal values (i.e., dual variables) which may themselves be functionally dependent on those force levels. This dependence of optimal strategies on force levels is best illustrated in the problem at hand by the strategy of X in which he splits his fire according to $\psi^* = \frac{b_2}{b_1+b_2}(1 - \frac{q_2Y_2}{p_2X_2})$ while Y is concentrating all of his fire on X's supporting elements, $\phi^*=0$. This strategy illustrates a case where the optimal allocation policy is not only dependent on force levels through the switching function, but is itself a function of two of those force levels.

This problem when compared to that presented by Kawara [5] illustrates that the nature of the attrition process in a differential game combat model has significant impact on optimal strategies. Again this is best illustrated here by the appearance of an extremal strategy in which X splits his fire between the two Y elements. This splitting of fire which does not appear in Kawara's problem was induced with only minor modification of Kawara's attrition

structure, and by allowing Y's primary force to attrite X's primary force. Because of this sensitivity of optimal strategies to changes in the attrition structure, care must be taken when attempting to model the combat decision making process in a differential game to insure that insights gained into optimal strategies are not biased by poorly constructed combat dynamics.

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Optimal fire-support strategies are studied through a deterministic differential game using Lanchester-type equations of warfare. In addition to the MAX-MIN principle, the theory of singular extremals is required to solve this prescribed duration combat problem. The combat is between two heterogeneous forces, each composed of infantry and a supporting weapon system (artillery). In contrast to previous

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work reported in the literature, the attrition structure of the problem at hand leads to the optimal fire-support strategy of the attacker requiring him to sometimes split his artillery fire between enemy infantry and artillery (counterbattery fire). Numerical examples are given. The military significance (based on the marginal value interpretation of the dual variables) of various optimality conditions is discussed.

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A problem of allocation of supporting fire in combat as a zero sum differential game.